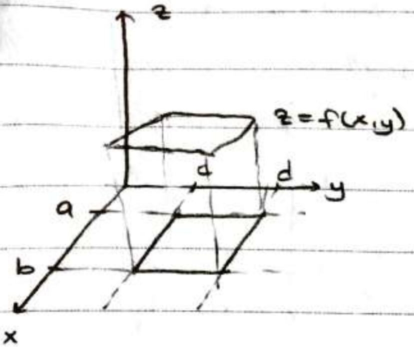
15.2 → DOUBLE INTEGRALS OVER GENERAL REGIONS

Last time:



$f(x,y) \geq 0$

$$\text{Volume} = \iint_{[a,b]\times[c,d]} f(x,y)\, dA$$

* Fubini If f is continuous on $[a,b] \times [c,d]$

$$\iint_{[a,b]\times[c,d]} f(x,y)\, dA = \int_{x=a}^{b} \left[\int_{c}^{d} f(x,y)\,dy \right] dx$$

$$= \int_{y=c}^{d} \left[\int_{a}^{b} f(x,y)\,dx \right] dy$$

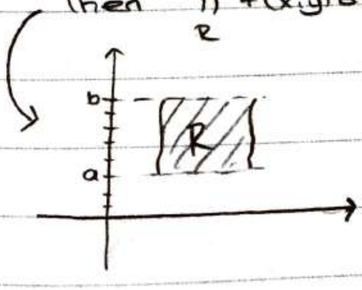
* Fubini Let $f(x,y)$ be continuous on a closed and bounded region R

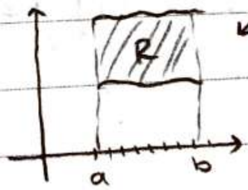
1. If $R = \{(x,y) : a \leq x \leq b \ , g_1(x) \leq y \leq g_2(x)\}$

   Then $\displaystyle\iint_R f(x,y)\, dA = \int_{x=a}^{b} \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$

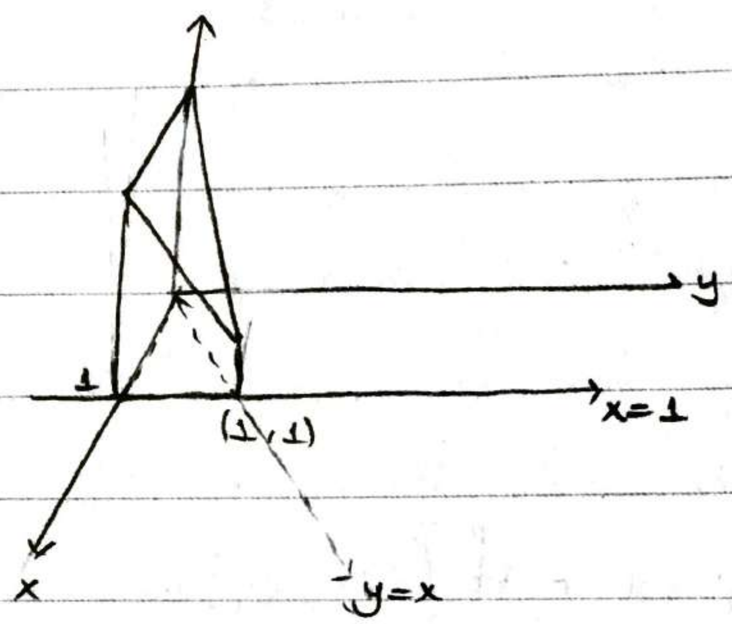
2. If $R = \{(x,y) : a \leq y \leq b \ , g_1(y) \leq x \leq g_2(y)\}$

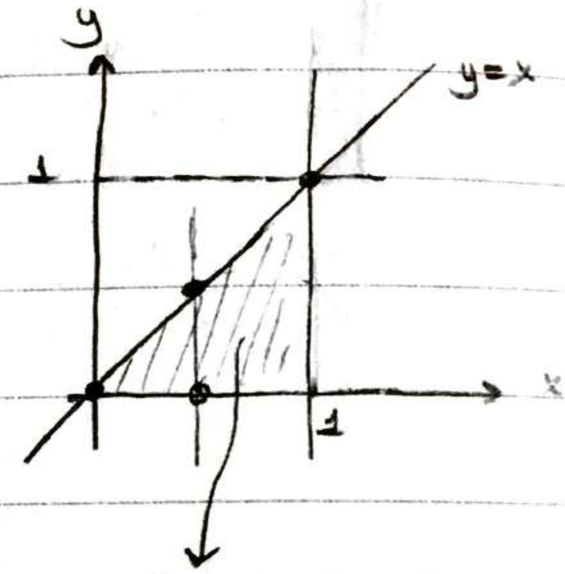
   Then $\displaystyle\iint_R f(x,y)\, dA = \int_{a}^{b} \left[\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx \right] dy$





ex: Find the volume of the prism whose base is a triangle in the xy-plane bounded by the x-axis and the lines $y=x$, $x=1$ and top in the plane $z=f(x,y)=3-x-y$





$0 \leq x \leq 1$

$0 \leq y \leq x$

volume: $\int_{x=0}^{1}\left[\int_{0}^{x}(3-x-y)\,dy\right]dx = \int_{0}^{1}\left[3y-xy-\frac{y^2}{2}\Big|_{y=0}^{x}\right]dx$

$= \int_{0}^{1}\left(3x-x^2-\frac{x^2}{2}\right)dx$

$= \frac{3x^2}{2}-\frac{x^3}{3}-\frac{x^3}{6}\Big|_{0}^{1} = 1$

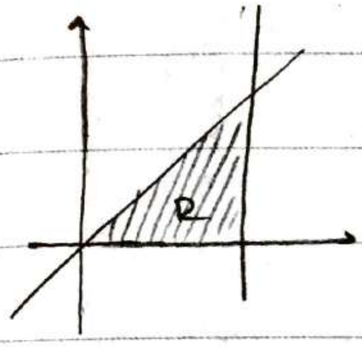
2nd method

$0 \leq y \leq 1$

$y \leq x \leq 1$

Volume $= \int_{y=0}^{1}\left[\int_{x=y}^{1}(3-x-y)\,dx\right]dy = 1$

ex: $\iint_R \frac{\sin x}{x}\, dA$ where R is the triangle in the xy-plane bounded by the line $y=x$ and the line $x=1$



$$\int_{y=0}^{1}\left[\int_{x=y}^{1} \frac{\sin x}{x}\, dx\right] dy = ?$$

$\int \frac{\sin x}{x}\, dx \rightarrow$ Not known.

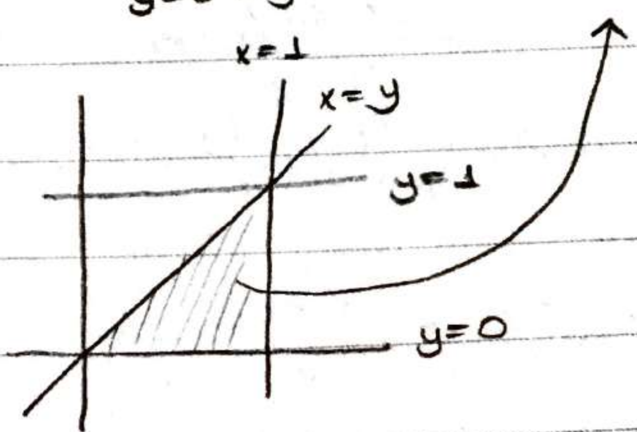
$$\int_{x=0}^{1}\left[\int_{y=0}^{x} \frac{\sin x}{x}\, dy\right] dx = \int_{x=0}^{1} \frac{\sin x}{x} \int_{y=0}^{x} 1\, dy \;\; dx$$

$$= \int_{x=0}^{1} \frac{\sin x}{x} \cdot y \Big|_0^x dx$$

$$= \int_{x=0}^{1} \frac{\sin x}{x} \cdot x \cdot dx$$

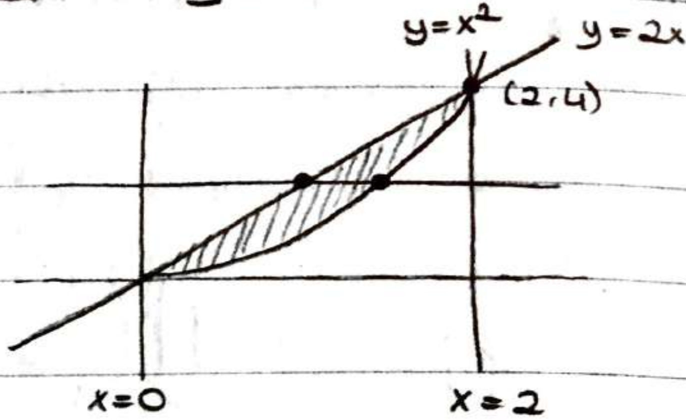
$$= \int_{x=0}^{1} \sin x\, dx = -\cos x \Big|_0^1 = \underline{-\cos x + 1}$$

ex: Find $\int_{y=0}^{1}\left[\int_{x=y}^{1} \frac{\sin x}{x}\, dx\right] dy = \int_{x=0}^{1}\left[\int_{y=0}^{x} \frac{\sin x}{x}\, dy\right] dx = \underline{1-\cos 1}$



ex: Change the order of integration

$$\int_0^2 \left[\int_{x^2}^{2x} (4x+2)\,dy \right] dx$$



$0 \le x \le 2$

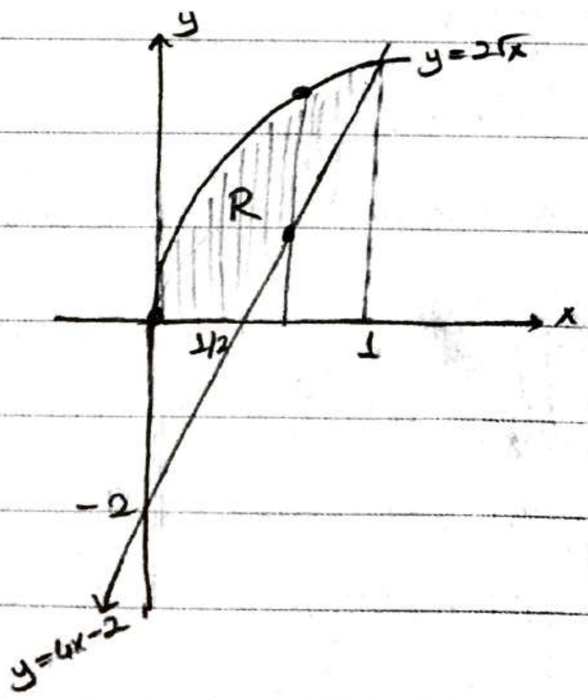
$x^2 \le y \le 2x$

$$\int_{y=0}^{4} \left[\int_{x=y/2}^{\sqrt{y}} (4x+2)\,dx \right] dy$$

$0 \le y \le 4$

$y/2 \le x \le \sqrt{y}$

ex: Find the volume of the wedgelike solid below the surface $z = 16 - x^2 - y^2$ above the region R bounded by $y = 2\sqrt{x}$, $y = 4x - 2$ and the x-axis.



1st method

* $$\text{Volume} = \int_{x=0}^{1/2} \left[\int_{y=0}^{2\sqrt{x}} (16 - x^2 - y^2)\,dy \right] dx$$

$$+ \int_{x=1/2}^{1} \left[\int_{y=4x-2}^{2\sqrt{x}} (16 - x^2 - y^2)\,dy \right] dx$$

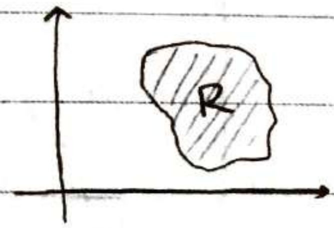
$$= 20803/1680$$

2nd method

* $$\text{Volume} = \int_{y=0}^{2} \left[\int_{x=y^2/4}^{\frac{y+2}{4}} (16 - x^2 - y^2)\,dx \right] dy$$

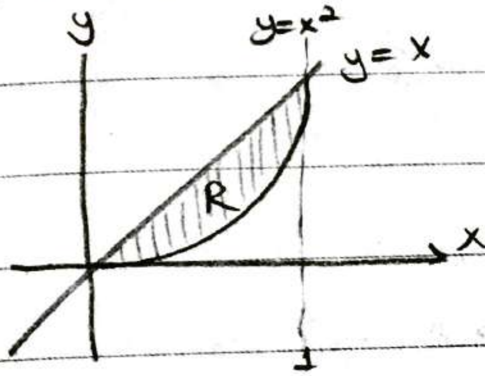
$$= 20803/1680$$

15.3 → AREA BY DOUBLE INTEGRATION



$$\text{Area}(R) = \iint_R 1 \cdot dA$$

ex: Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant



calculus 1: $\displaystyle\int_{x=0}^{1} [x - x^2]\, dx$

Calculus 2: $\displaystyle\iint_R 1\, dA = \int_{x=0}^{1}\left[\int_{y=x^2}^{x} 1\, dy\right] dx$

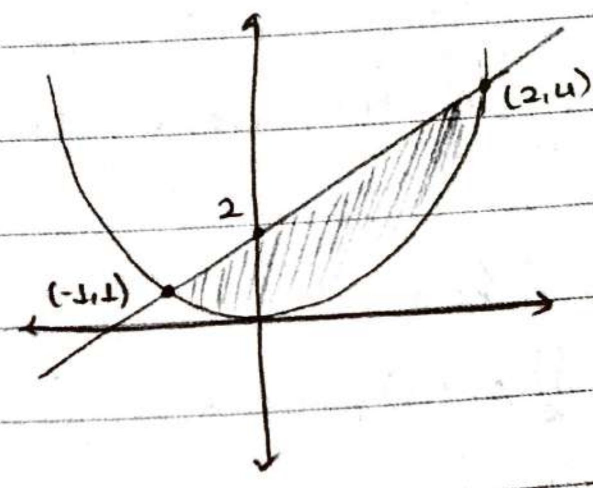
$\displaystyle = \int_{y=0}^{1}\left[\int_{y}^{\sqrt{y}} 1\, dx\right] dy$

R: $0 \le x \le 1$

$x^2 \le y \le x$

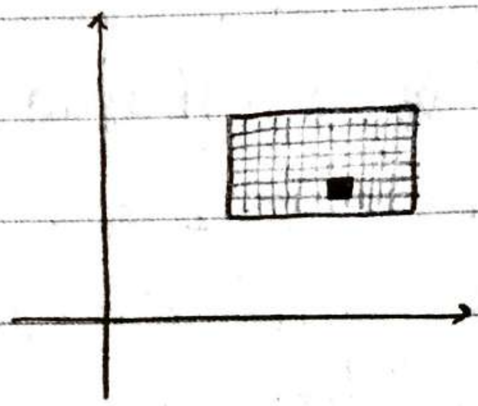
$\rightarrow 1/2 - 1/3 = 1/6$

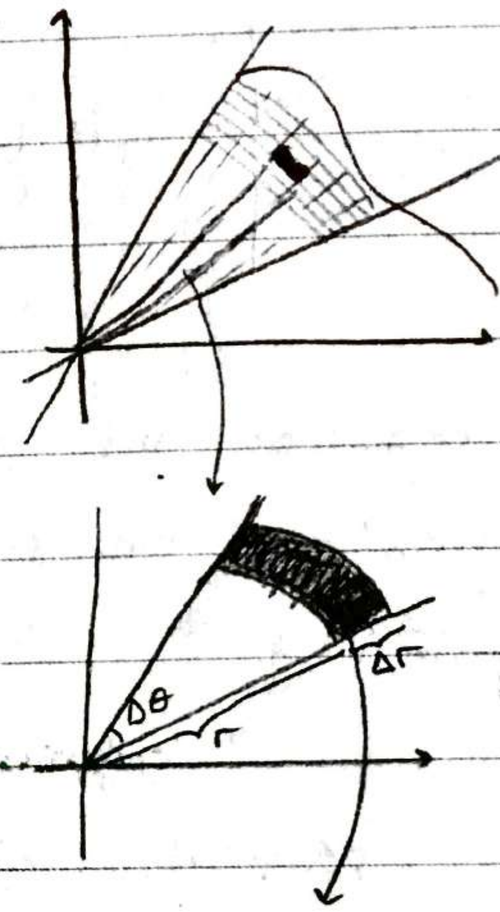
ex! Find the area of the region enclosed by $y = x^2$ and $y = x + 2$



Area: $\displaystyle\int_{x=-1}^{2}\left[\int_{y=x^2}^{x+2} 1\, dy\right] dx$

15.4 → DOUBLE INTEGRALS IN POLAR FORM





Area: $\pi(r+\Delta r)^2 \frac{\Delta\theta}{2\pi} - \pi r^2 \frac{\Delta\theta}{2\pi}$

$$= \frac{\Delta\theta}{2}\left[\cancel{r^2}+2r\Delta r+(\Delta r)^2-\cancel{r^2}\right]$$

$$= \frac{\Delta\theta}{2}\left(2r\Delta r+(\Delta r)^2\right.$$

(Δr very small)

Ignore $(\Delta r)^2$

* Area: $\approx \frac{\Delta\theta}{2} 2r\Delta r = r\Delta\theta\Delta r$

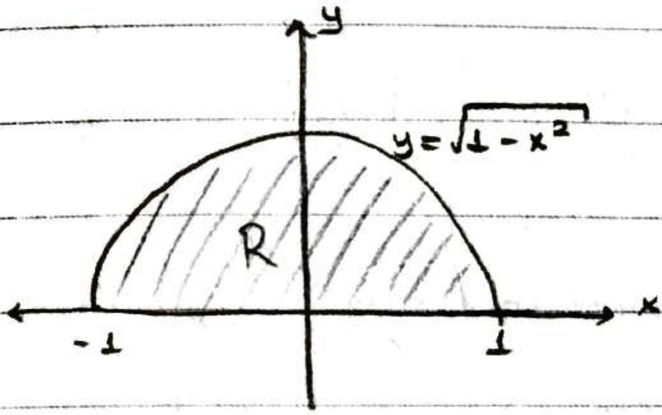
* Volume: $\approx \sum_{i=0}^{n}\sum_{j=0}^{m} \underbrace{f(r_i,\theta_j)}_{\text{height}} \underbrace{r_i\,\Delta\theta_j\,\Delta r}_{\text{area}}$

$\downarrow$ Limit

* Volume $= \iint_R f(r,\theta)\, r\, dr\, d\theta$

ex: Evaluate

$\iint_R e^{x^2+y^2}\,dy\,dx$ where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$

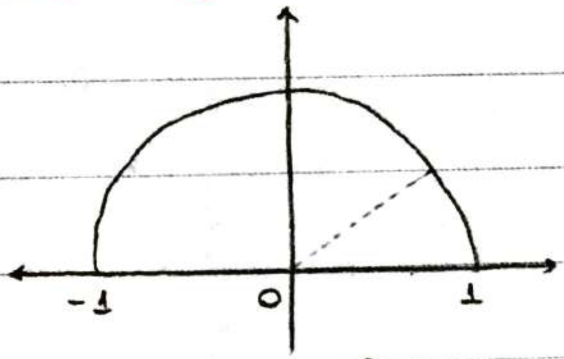


R: $-1 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$

$$\int_{x=-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx \longrightarrow \text{can not find this}$$

2nd way Polar coordinates.



$0 \le \theta \le \pi$

$0 \le r \le 1$

$x^2+y^2=r^2$

Integral: $\displaystyle\int_{\theta=0}^{\pi}\int_{r=0}^{1} e^{r^2}\cdot \underline{\underline{r}}\cdot dr\,d\theta$

$$\int_{r=0}^{1} e^{r^2}r\,dr = \int_{u=0}^{1} e^u\,du/2 = \frac{e^u}{2}\Big|_0^1 = \frac{e-1}{2}$$

$r^2=u$

$2r\,dr=du$

$$\int_{\theta=0}^{\pi}\left(\frac{e-1}{2}\right)d\theta = \frac{e-1}{2}\,\theta\Big|_0^{\pi}$$

$$= \frac{e-1}{2}\cdot\pi$$

ex: Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$$

1st way!

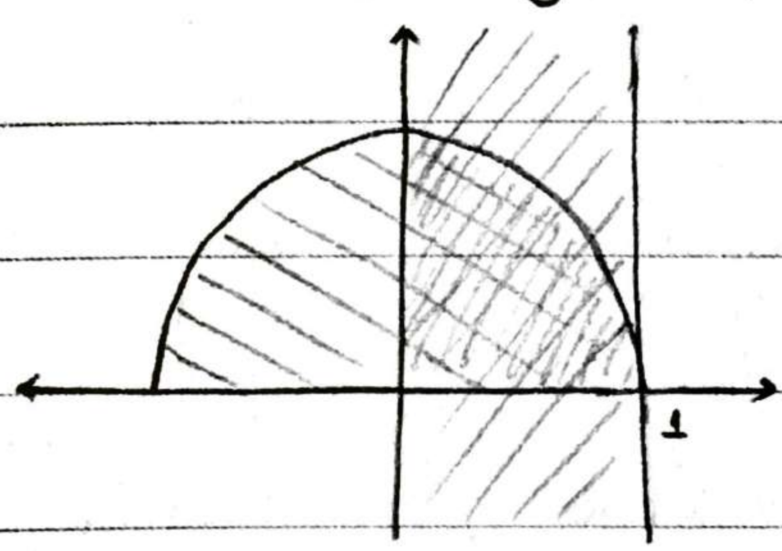
$$\int_0^1 \left[x^2 y + \frac{y^3}{3} \Big|_0^{\sqrt{1-x^2}} \right] dx$$

$$= \int_0^1 \left[x^2\sqrt{1-x^2} + \frac{(1-x^2)\sqrt{1-x^2}}{3} \right] dx$$

Doable use $x = \sin\theta$ transformation but hard !!

2nd way! $0 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$



$0 \le \theta \le \pi/2$

$0 \le r \le 1$

$$\text{Integral} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2\, \underline{r}\, dr\, d\theta$$

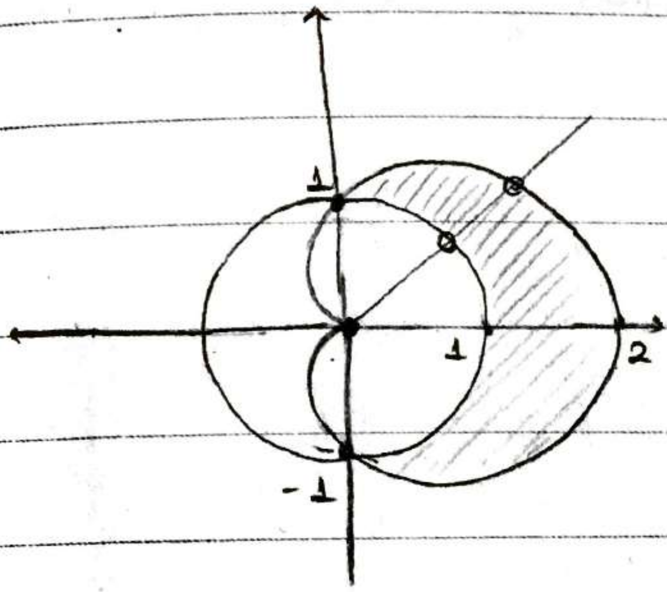
$$= \int_{\theta=0}^{\pi/2} \frac{r^4}{4} \Big|_{r=0}^{1} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{4}\, d\theta$$

$$= \frac{\theta}{4} \Big|_0^{\pi/2} = \frac{\pi}{8}$$

(15.4 → 28)

ex: Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$



$-\pi/2 \leq \theta \leq \pi/2$

$1 \leq r \leq 1 + \cos\theta$

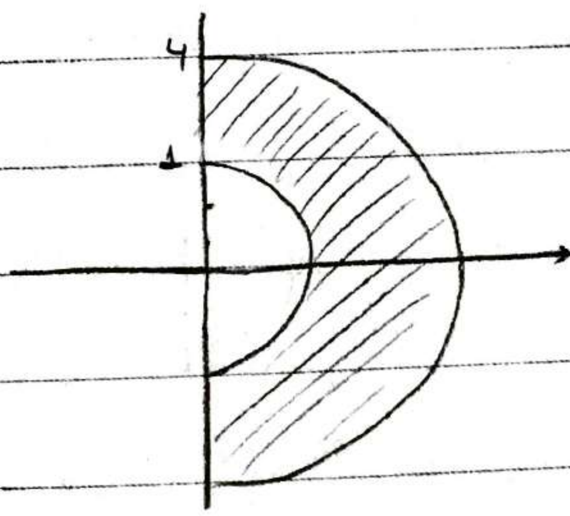
$$\text{Area: } \int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} 1 \cdot r \, dr \, d\theta$$

$$= \int_{\pi/2}^{-\pi/2} \frac{r^2}{2} \Big|_{1}^{1+\cos\theta} d\theta$$

$$= -\frac{1}{2} \int_{-\pi/2}^{\pi/2} \left[(1+\cos\theta)^2 - 1 \right] d\theta$$

(15.4 → 2)

ex: Describe the region in polar coordinates



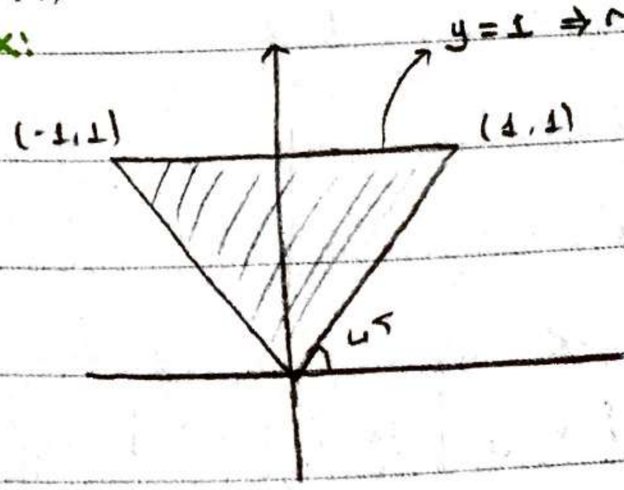
$-\pi/2 \leq \theta \leq \pi/2$

$1 \leq r \leq 4$

(15.4 → 3)

ex:

$y = 1 \Rightarrow r\sin\theta = 1 \quad r = 1/\sin\theta = \csc\theta$

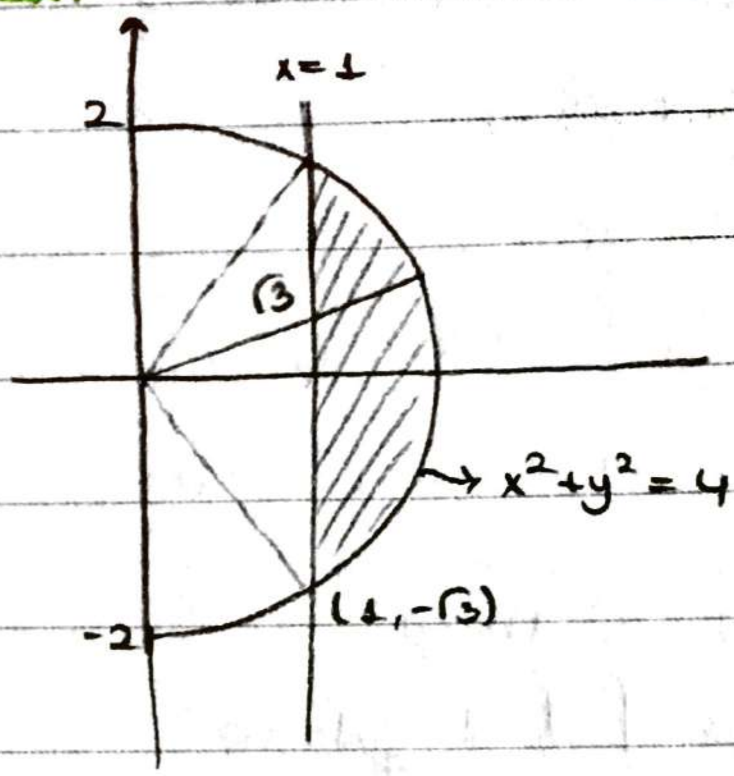


$\pi/4 \leq \theta \leq 3\pi/4$

$0 \leq r \leq \frac{1}{\sin\theta}$

(15.4→6)

ex:



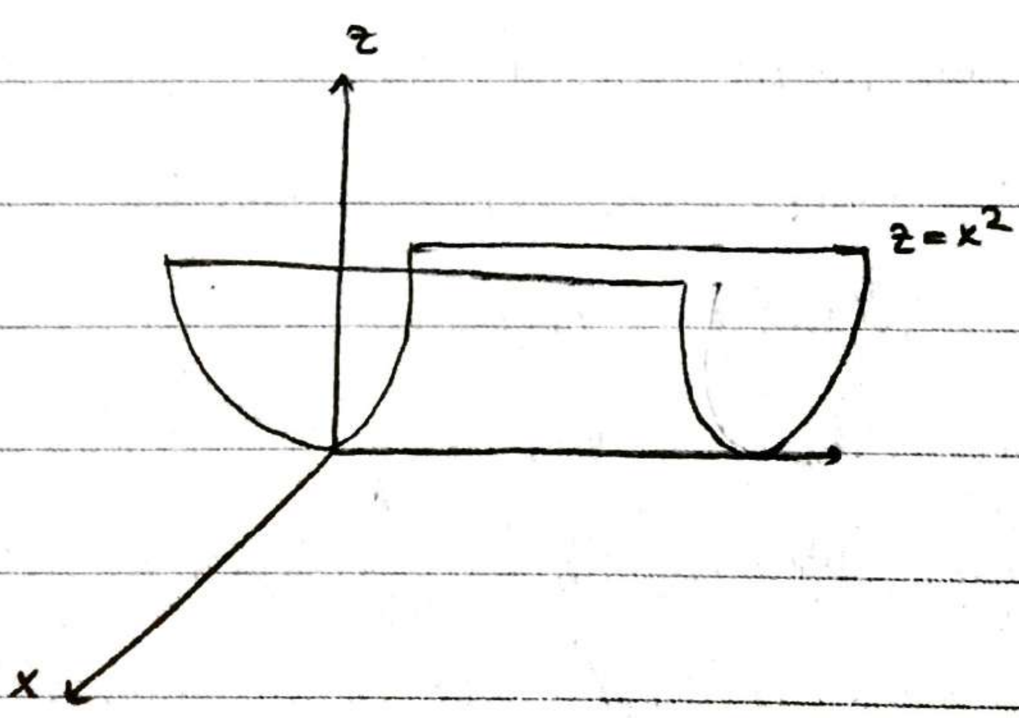
$\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

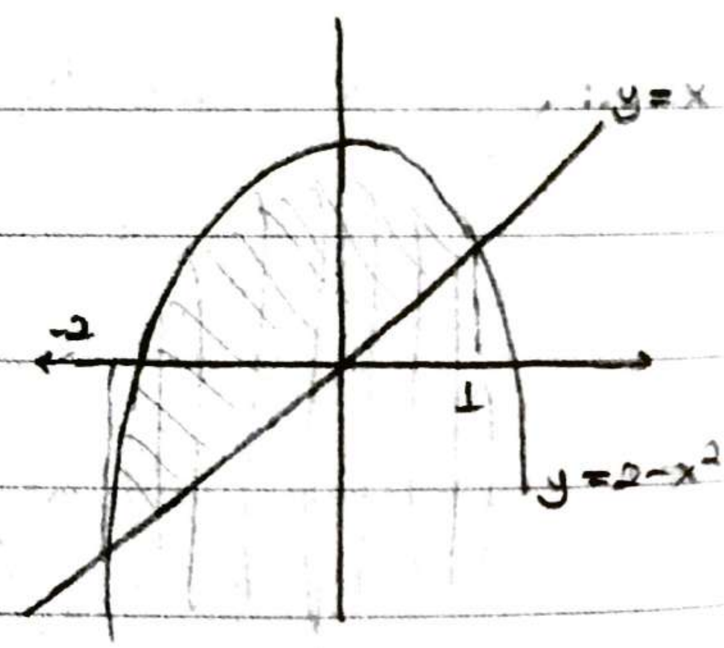
$-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$

$1/\cos\theta \le r \le 2$

(15.2→58)

ex: Find the volume of region bounded above by the cylinder $z=x^2$ and below by the region enclosed by the paropola $y=2-x^2$ and the line $y=x$ in the xy-plane.





$x=2-x^2$

$x=-2 \quad x=1$

$-2 \le x \le 1 \quad \Leftarrow R$

$x \le y \le 2-x^2$

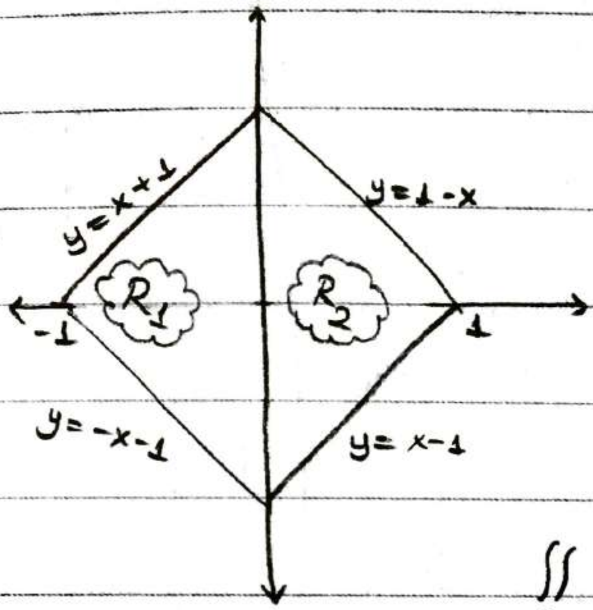
$$\text{Volume} = \iint_R x^2 \, dA$$

$$= \int_{x=-2}^{1} \left[\int_{x}^{2-x^2} x^2 dy \right] dx = \int_{-2}^{1} x^2 y \Big|_{x}^{2-x^2} dx$$

$$= \int_{-2}^{1} \left[x^2(2-x^2) - x^2 x \right] dx$$

(15.2→55)

ex: $\iint_R (y-2x^2)\,dA$ where R is the region bounded by the square $|x|+|y|=1$



$R_1: -1 \le x \le 0$

$-x-1 \le y \le x+1$

$R_2: 0 \le x \le 1$

$x-1 \le y \le 1-x$

$$\iint_R (y-2x^2)\,dA = \iint_{R_1} (y-2x^2)\,dA + \iint_{R_2} (y-2x^2)\,dA$$

$$= \int_{-1}^{0}\int_{-x-1}^{x+1} (y-2x^2)\,dy\,dx + \int_{0}^{1}\int_{x-1}^{1-x} (y-2x^2)\,dy\,dx$$